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# Level-rank duality of D-branes on the $SU(N)$ group manifold

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## Abstract

The consequences of level-rank duality for untwisted D-branes on an  $SU(N)$  group manifold are explored. Relations are found between the charges of D-branes (which are classified by twisted K-theory) belonging to  $\widehat{\mathfrak{su}}(N)_K$  and  $\widehat{\mathfrak{su}}(K)_N$  WZW theories, in the case of odd  $N + K$ . An isomorphism between the charge algebras is also demonstrated in this case.

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# 1 Introduction

Understanding the nature of D-branes is a central issue of contemporary string theory, particularly the properties of D-branes in nontrivial gravitational and  $B$ -field backgrounds. (For a review, see ref. [1].) One approach to this question is to study D-branes on group manifolds [2]–[21], where the background is highly symmetric, and the associated conformal field theory (the WZW model) exactly solvable. The D-branes in this theory correspond to boundary states of the WZW model, which can be studied algebraically. For the D-branes to be stable, the bosonic WZW theory should be regarded as part of a supersymmetric theory on the group manifold [12, 14]; we will only consider the simplest case where the boson and fermion sectors are decoupled. Strings on arbitrary group manifolds can also serve as building blocks of coset models.

Much can be learned about D-branes by studying their charges, which are classified by K-theory or, in the presence of a cohomologically nontrivial  $H$ -field background, twisted K-theory [22]. The charge group for D-branes on a simply-connected group manifold  $G$  with level  $K$  is given by the twisted K-group [12, 14, 23, 24, 25, 19]

$$K^*(G) = \oplus_{i=1}^m \mathbb{Z}_x, \quad m = 2^{\text{rank } G - 1} \quad (1.1)$$

where  $\mathbb{Z}_x \equiv \mathbb{Z}/x\mathbb{Z}$  with  $x$  an integer depending on  $G$  and  $K$ . For  $\widehat{\text{su}}(N)_K$ , for example,  $x$  is given by [12]

$$x_{N,K} \equiv \frac{N + K}{\text{gcd}\{N + K, \text{lcm}\{1, \dots, N - 1\}\}}. \quad (1.2)$$

One of the  $\mathbb{Z}_x$  factors in the charge group corresponds to the charge of untwisted (symmetry-preserving) D-branes. The remaining factors (for  $N > 2$ ) correspond to the charges of other branes of the theory [12, 14, 19].

An intriguing aspect of WZW models is level-rank duality, a relationship between various quantities in the  $\widehat{\text{su}}(N)_K$  model and corresponding quantities in the  $\widehat{\text{su}}(K)_N$  model [26]–[28]. Similar dualities occur for orthogonal and symplectic groups, and also in Chern–Simons theories [27]–[29]. Heretofore no analysis of level-rank duality has been given for boundary WZW theories. In this paper, we begin the study of this issue by considering the relationship between untwisted D-branes on  $\widehat{\text{su}}(N)_K$  and  $\widehat{\text{su}}(K)_N$  group manifolds.

In section 2, we review some necessary details of the WZW theory without boundary, including simple current symmetries and level-rank duality. Section 3 contains a description of untwisted D-branes of the  $\widehat{\text{su}}(N)_K$  WZW theory with boundary; these D-branes are labelled by irreducible representations  $\lambda$  of  $\text{su}(N)$  that correspond to integrable highest-weight representations of  $\widehat{\text{su}}(N)_K$ . In section 4, we demonstrate the relationship between the charge  $Q_\lambda$  of an  $\widehat{\text{su}}(N)_K$  D-brane and that of a related D-brane of  $\widehat{\text{su}}(K)_N$ . Specifically, we find

$$\tilde{Q}_{\tilde{\lambda}} = (-1)^{r(\lambda)} Q_\lambda \pmod{x}, \quad \text{for } N + K \text{ odd} \quad (1.3)$$

where  $\tilde{Q}_{\tilde{\lambda}}$  is the charge of the  $\widehat{\text{su}}(K)_N$  D-brane labelled by the representation  $\tilde{\lambda}$  of  $\text{su}(K)$  obtained by transposing the Young tableau of  $\lambda$ ,  $r(\lambda)$  is the number of boxes of the Young tableau  $\lambda$ , and  $x = \min\{x_{N,K}, x_{K,N}\}$ . (A similar but more complicated relationship is expected when  $N + K$  is even.<sup>3</sup>) We also show that the charge algebra of D-branes is

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<sup>3</sup>Note added: see ref. [30].

isomorphic to the charge algebra of dual D-branes in the level-rank dual WZW model (for  $N+K$  odd), and that the energies of level-rank dual D-branes are equal. Concluding remarks constitute section 5.

## 2 Bulk WZW Theory and level-rank duality

Strings on group manifolds are described by the Wess-Zumino-Witten conformal field theory. In this section, we review some aspects of the WZW model, including simple currents and level-rank duality, that will be needed in subsequent sections to understand D-branes on group manifolds.

The WZW model is a rational conformal field theory whose chiral algebra (for both left- and right-movers) is the (untwisted) affine Lie algebra  $\hat{g}_K$  at level  $K$  with Virasoro central charge  $c = K \dim g / (K + h^\vee)$ , where  $g$  is the finite Lie algebra associated with  $\hat{g}_K$ , and  $h^\vee$  the dual Coxeter number of  $g$ . The building blocks of the WZW model are integrable highest-weight representations  $V_\lambda \in P_+^K$  of  $\hat{g}_K$ , which are labelled by  $\lambda$ , the irreducible representation of  $g$  spanning the lowest-conformal-weight-subspace of the  $\hat{g}_K$  representation. Associated with the affine Lie algebra  $\hat{g}_K$  is an extended Dynkin diagram, which has one more dot than the Dynkin diagram of  $g$ . Correspondingly, the highest-weight representation  $V_\lambda$  possesses an extra Dynkin index

$$a_0 = K - \sum_{i=1}^n m_i a_i, \quad n = \text{rank } g, \quad (2.1)$$

where  $a_i$  are the Dynkin indices of  $\lambda$  and the integers  $m_i$  are the components of the highest co-root. (For  $\widehat{\text{su}}(N)_K$ , the affine Lie algebra with which we will be primarily concerned,  $m_i = 1$  for all  $i$ .) An *integrable* highest-weight representation is one satisfying  $a_0 \geq 0$ .

For purposes of discussing level-rank duality, it is useful to describe irreducible representations of  $g$  in terms of Young tableaux. For example, an irreducible representation of  $\text{su}(N)$  with Dynkin indices  $a_i$  corresponds to a Young tableau with  $N-1$  or fewer rows, with row lengths

$$\ell_i = \sum_{j=i}^{N-1} a_j, \quad i = 1, \dots, N-1. \quad (2.2)$$

Let  $r(\lambda) = \sum_i \ell_i$  denote the number of boxes of the tableau. Representations  $\lambda$  corresponding to integrable highest-weight representations  $V_\lambda$  of  $\widehat{\text{su}}(N)_K$  obey  $\ell_1(\lambda) \leq K$ , *i.e.*, their Young tableaux have  $K$  or fewer columns.

The set of affine characters of the integrable highest-weight representations

$$\chi_\lambda(\tau) = \text{Tr}_{V_\lambda} e^{2\pi i \tau (L_0 - c/24)}, \quad \lambda \in P_+^K \quad (2.3)$$

is closed under the modular transformation  $\tau \rightarrow -1/\tau$ , the mixing being described by the modular transformation matrix  $S_{\mu\nu}$ .

A primary field  $\phi_\lambda$  of the conformal field theory is associated with each integrable highest-weight representation  $V_\lambda$ . The multiplicities of the primary fields appearing in the operator

product expansion of a pair of primary fields of the WZW model are given by the fusion coefficients  $N_{\mu\nu}^\lambda$  appearing in the fusion algebra

$$\phi_\mu \cdot \phi_\nu = \sum_{\lambda \in P_+^K} N_{\mu\nu}^\lambda \phi_\lambda \quad (2.4)$$

where  $N_{\mu\nu}^\lambda$  is given by Verlinde's formula [31]

$$N_{\mu\nu}^\lambda = \sum_{\rho \in P_+^K} \frac{S_{\mu\rho} S_{\nu\rho} S_{\lambda\rho}^*}{S_{0\rho}} , \quad (2.5)$$

with 0 denoting the identity representation. For fixed  $\mu$  and  $\nu$ , and for  $K$  sufficiently large, the fusion coefficients  $N_{\mu\nu}^\lambda$  become equal to  $\overline{N}_{\mu\nu}^\lambda$ , the multiplicities appearing in the tensor product decomposition of representations of  $g$

$$\mu \otimes \nu = \bigoplus_{\lambda} \overline{N}_{\mu\nu}^\lambda \lambda . \quad (2.6)$$

In the case of  $\widehat{\mathfrak{su}}(N)_K$ , “sufficiently large” means  $\ell_1(\mu) + \ell_1(\nu) \leq K$ , and  $\overline{N}_{\mu\nu}^\lambda$  are just the Littlewood-Richardson coefficients.

In this paper, we only consider the WZW theory with a diagonal closed-string spectrum:

$$\mathcal{H}^{\text{closed}} = \bigoplus_{\lambda \in P_+^K} V_\lambda \otimes \overline{V}_{\lambda^*} \quad (2.7)$$

where  $\overline{V}$  represents right-moving states, and  $\lambda^*$  denotes the representation conjugate to  $\lambda$ , *i.e.*, such that  $N_{\lambda\lambda^*}^0 = 1$ . The partition function for this theory

$$Z(\tau) = \sum_{\lambda \in P_+^K} |\chi_\lambda(\tau)|^2 \quad (2.8)$$

is automatically modular invariant.

### Simple current symmetries

Automorphisms of the extended Dynkin diagram shuffle the Dynkin indices and thus relate different integrable representations to one another. For example, the extended Dynkin diagram of  $\widehat{\mathfrak{su}}(N)_K$  has a  $\mathbb{Z}_N$  symmetry  $\sigma$  which takes a representation  $\lambda$  into  $\lambda' = \sigma(\lambda)$ , whose Dynkin indices are  $a'_i = a_{i-1}$  for  $i = 1, \dots, N-1$ , and  $a'_0 = a_{N-1}$ . The Young tableau for  $\lambda'$  is obtained by placing a row of length  $K$  on top of the tableau for  $\lambda$ , and deleting any columns of length  $N$  that may result. The modular transformation matrix for  $\widehat{\mathfrak{su}}(N)_K$  transforms under  $\sigma$  as [28]

$$S_{\sigma(\mu)\nu} = e^{-2\pi i r(\nu)/N} S_{\mu\nu} , \quad (2.9)$$

as a result of which the fusion rule coefficients for  $\widehat{\mathfrak{su}}(N)_K$  satisfy

$$N_{\sigma^m(\mu)\sigma^n(\nu)}^{\sigma^l(\lambda)} = N_{\mu\nu}^\lambda \quad \text{for } m+n = l \bmod N . \quad (2.10)$$

The orbits  $\{\lambda, \sigma(\lambda), \dots, \sigma^{N-1}(\lambda)\}$  of integrable highest-weight representations are termed “cominimal equivalence classes.” (Some orbits may have fewer elements.) The members of

the orbit of the identity representation  $\sigma^j(0)$ , with  $j = 1, \dots, N-1$ , are termed “cominimal representations” or “simple currents,” and correspond to rectangular Young tableaux with  $j$  rows and  $K$  columns. The fusion algebra for these representations

$$\phi_\lambda \cdot \phi_{\sigma^j(0)} = \phi_{\sigma^j(\lambda)} \quad (2.11)$$

contains only one term by virtue of eq. (2.10).

### Level-rank duality

An intriguing relation, level-rank duality, exists between the WZW model for  $\widehat{\mathfrak{su}}(N)_K$  and the corresponding WZW model with  $N$  and  $K$  exchanged [26]-[29]. The Young tableau  $\lambda$  corresponding to an integrable highest-weight representation of  $\widehat{\mathfrak{su}}(N)_K$  maps under transposition (*i.e.*, exchange of rows and columns) to a Young tableau  $\tilde{\lambda}$  that corresponds to an integrable highest-weight representation of  $\widehat{\mathfrak{su}}(K)_N$  (possibly after removing any columns of length  $K$ ). This map is not one-to-one, since cominimally-equivalent representations of  $\widehat{\mathfrak{su}}(N)_K$  may map into the same representation of  $\widehat{\mathfrak{su}}(K)_N$  (due to the removal of columns). It is clear, however, that the cominimal equivalence classes of the two theories *are* in one-to-one correspondence.

The modular transformation matrices and fusion rule coefficients of the  $\widehat{\mathfrak{su}}(N)_K$  theory obey simple relations under the exchange of  $N$  and  $K$ . Letting  $S_{\mu\nu}$  and  $\tilde{S}_{\tilde{\mu}\tilde{\nu}}$  denote the modular transformation matrices of  $\widehat{\mathfrak{su}}(N)_K$  and  $\widehat{\mathfrak{su}}(K)_N$ , one finds [28]

$$S_{\mu\nu} = \sqrt{\frac{K}{N}} e^{-2\pi i r(\mu)r(\nu)/NK} \tilde{S}_{\tilde{\mu}\tilde{\nu}}^* . \quad (2.12)$$

From this and eq. (2.5), it follows that [28]

$$N_{\mu\nu}^\lambda = \tilde{N}_{\tilde{\mu}\tilde{\nu}}^{\sigma^\Delta(\tilde{\lambda})}, \quad \Delta = \frac{r(\mu) + r(\nu) - r(\lambda)}{N} \in \mathbb{Z} \quad (2.13)$$

where  $\tilde{N}$  denotes the fusion rule multiplicities of  $\widehat{\mathfrak{su}}(K)_N$ . (For  $N$  sufficiently large — *viz.*, for  $N > k_1(\mu) + k_1(\nu)$ , where  $k_1(\mu)$  denotes the length of the first column of  $\mu$  —  $\Delta$  vanishes, so that on the right-hand side of the fusion algebra (2.4),  $\lambda$  is simply dual to  $\tilde{\lambda}$ , its transpose, but in general the relation is more complicated.)

## 3 Boundary WZW Theory and D-brane charges

D-branes on group manifolds have received a lot of attention, from both the algebraic and geometric point of view [2]-[21]. Algebraically, D-branes on group manifolds can be studied in terms of the possible boundary conditions that can be imposed on a WZW model with boundary. Let the open string world-sheet be the upper half plane. The nonvanishing components of the stress-energy tensor must satisfy  $T(z) = \overline{T}(\bar{z})$  on the boundary  $z = \bar{z}$ . Additional restrictions may be imposed on the currents of the affine Lie algebra on the boundary, *e.g.*

$$\left[ J^a(z) - \omega \overline{J}^a(\bar{z}) \right] \Big|_{z=\bar{z}} = 0, \quad (3.1)$$

where  $\omega$  is an automorphism of the affine Lie algebra, although eq. (3.1) is not required by the conformal symmetry. Open-closed string duality correlates the boundary conditions (3.1) of the boundary WZW model with certain coherent states  $|B\rangle\rangle$  of the bulk WZW model satisfying

$$[J_n^a + \omega \bar{J}_{-n}^a] |B\rangle\rangle = 0 \quad (3.2)$$

where  $J_n^a$  are the modes of the current algebra generators.

Symmetry-preserving, or untwisted, D-branes correspond to  $\omega = 1$ , and it is this special class of branes that will be the focus of this paper. For  $\omega = 1$ , equation (3.2) can be satisfied by a state belonging to a single sector  $V_\lambda \otimes \bar{V}_{\lambda^*}$  of the WZW theory; such states are termed “Ishibashi states”  $|\lambda\rangle\rangle_I$  [32]. Because we are considering the diagonal WZW theory, all states  $|\lambda\rangle\rangle_I$  for  $\lambda \in P_+^K$  belong to the spectrum of the bulk WZW theory.

Boundary states corresponding to D-branes must satisfy additional (Cardy) conditions [33], whose solution may be written as certain linear combinations of the Ishibashi states known as Cardy states  $|\lambda\rangle\rangle_C$ . For untwisted D-branes of the diagonal WZW theory, these states are of the form

$$|\lambda\rangle\rangle_C = \sum_{\mu \in P_+^K} \frac{S_{\lambda\mu}}{\sqrt{S_{0\mu}}} |\mu\rangle\rangle_I. \quad (3.3)$$

From a geometric point of view (and for large values of  $K$ ), the untwisted D-branes wrap conjugacy classes on the group manifold and are stabilized by flux stabilization [4]–[14]. Quantization imposes constraints on the allowed conjugacy classes, which are in one-to-one correspondence with integrable highest-weight representations  $V_\lambda \in P_+^K$  of  $\hat{g}_K$ .

## D-brane charges

Next we consider the charge of the untwisted D-brane associated with the boundary state  $|\lambda\rangle\rangle_C$ . The state  $|0\rangle\rangle_C$  corresponds geometrically to a D0-brane located at the identity element of the group manifold, to which we assign unit charge,  $Q_0 = 1$ . A collection of  $n$  such D0-branes has D0-charge  $n$ . By renormalization group flow arguments presented in refs. [34, 7, 10, 11, 12, 14], these D0-branes may form a bound state (D-brane) associated with the Cardy state  $|\lambda\rangle\rangle_C$ , corresponding to an  $n$ -dimensional representation  $\lambda$  of  $g$ . Hence the D0-charge  $Q_\lambda$  of this D-brane is  $(\dim \lambda)$ , but is only defined modulo some integer  $x$  [11, 12, 14, 17].

By considering condensation of D-branes, one finds that the charges must obey the fusion algebra [12]

$$Q_\mu \cdot Q_\nu = \sum_{\lambda \in P_+^K} N_{\mu\nu}^\lambda Q_\lambda. \quad (3.4)$$

From eq. (2.6), one has

$$(\dim \mu)(\dim \nu) = \sum_\lambda \bar{N}_{\mu\nu}^\lambda (\dim \lambda) \geq \sum_{\lambda \in P_+^K} N_{\mu\nu}^\lambda (\dim \lambda). \quad (3.5)$$

For sufficiently large  $K$ , the last inequality is saturated, in which case eq. (3.4) is consistent with  $Q_\lambda = \dim \lambda$ . In general, however, the charge algebra (3.4) is only satisfied modulo  $x$ , which is the largest integer for which

$$(\dim \mu)(\dim \nu) = \sum_{\lambda \in P_+^K} N_{\mu\nu}^\lambda (\dim \lambda) \quad \text{mod } x \quad (3.6)$$

holds for all  $\mu, \nu \in P_+^K$ .

Now we turn to the specific case of  $\widehat{\mathfrak{su}}(N)_K$ . To determine the value of  $x$ , it is sufficient [12] to consider the fusion algebra (2.11) involving simple currents and the fundamental representations  $\Lambda_s$  of  $\mathfrak{su}(N)$ ,  $s = 1, \dots, N-1$ , whose Young tableaux are  $\boxplus_s$ . This implies

$$\begin{aligned} (\dim \Lambda_s)(\dim \sigma(0)) &= \dim \sigma(\Lambda_s) \mod x, \\ \binom{N}{s} \binom{N+K-1}{K} &= \frac{(N+K-1)!}{(N-s)!(K-1)!s!(K+s)} \mod x. \end{aligned} \quad (3.7)$$

Then  $x$  is given by greatest common denominator of the difference between the two sides of eq. (3.7)

$$x = \gcd \left\{ \frac{s}{K+s} \binom{N+K}{K} \binom{N}{s} \mid s = 1, \dots, N-1 \right\}. \quad (3.8)$$

In refs. [12, 17], it is shown that eq. (3.8) implies  $x = x_{N,K}$  where

$$x_{N,K} \equiv \frac{N+K}{\gcd\{N+K, \text{lcm}\{1, \dots, N-1\}\}}. \quad (3.9)$$

Hence, the charge algebra (3.4) is satisfied provided the charges of the untwisted D-branes in  $\widehat{\mathfrak{su}}(N)_K$  are defined modulo  $x_{N,K}$ ,

$$Q_\lambda = \dim \lambda \mod x_{N,K} \quad \text{for } \widehat{\mathfrak{su}}(N)_K. \quad (3.10)$$

(The values of  $x$  for all other simple groups is given in ref. [17].) Thus untwisted D-branes correspond to the first factor  $\mathbb{Z}_{x_{N,K}}$  of the twisted K-theory group (1.1).

Finally, we consider the relation between the charges of  $\widehat{\mathfrak{su}}(N)_K$  D-branes corresponding to the cominimally-equivalent Cardy states  $|\lambda\rangle\rangle_C$  and  $|\sigma(\lambda)\rangle\rangle_C$ . The action of  $\sigma$  on integrable highest-weight representations corresponds geometrically to the action of the center of  $g$  on the conjugacy classes of  $g$  [11, 14]. The rotation of a conjugacy classes, however, cannot change the (magnitude of the) charge of the associated brane, thus [14]

$$Q_{\sigma(\lambda)} = (-1)^{N-1} Q_\lambda \mod x_{N,K} \quad \text{for } \widehat{\mathfrak{su}}(N)_K \quad (3.11)$$

where the relative sign comes from the action of  $N-1$  elements of the Weyl group, each reflection changing the orientation of the D-brane.

## 4 Level-rank duality of WZW D-branes

In this section, we will establish a relation between the charges of untwisted D-branes of  $\widehat{\mathfrak{su}}(N)_K$  and those of the level-rank dual theory  $\widehat{\mathfrak{su}}(K)_N$ .

### Level-rank duality of D-brane charges

Level-rank duality relates the cominimal equivalence classes of  $\widehat{\mathfrak{su}}(N)_K$  to those of  $\widehat{\mathfrak{su}}(K)_N$ . Since untwisted D-branes are labelled by integrable highest-weight representations, and since by eq. (3.11) their charges are invariant (modulo sign and modulo  $x$ ) under the operation

$\sigma$ , and therefore depend only on the cominimal equivalence class of the representation, it would be reasonable to expect the charges of level-rank dual D-branes to be equal (modulo sign and modulo  $x$ ). Indeed we will demonstrate below that this expectation is borne out, provided  $N + K$  is odd. The situation is unclear for  $N + K$  even.

Since the charges of  $\widehat{\text{su}}(N)_K$  D-branes are only defined modulo  $x_{N,K}$ , and the charges of  $\widehat{\text{su}}(K)_N$  D-branes are only defined modulo  $x_{K,N}$ , comparisons of these charges is only possible modulo  $\text{gcd}\{x_{N,K}, x_{K,N}\}$ . For the remainder of this section, we assume without loss of generality that  $N \geq K$ , in which case  $\text{gcd}\{x_{N,K}, x_{K,N}\} = x_{N,K}$ . (Thus the group with the larger rank has the smaller periodicity.) All D-brane charges will be considered modulo  $x \equiv x_{N,K}$ .

The relation (which we prove below) between  $Q_\lambda$ , the charge of the D-brane corresponding to the state  $|\lambda\rangle_C$  of  $\widehat{\text{su}}(N)_K$ , and  $\tilde{Q}_{\tilde{\lambda}}$ , the charge of the level-rank dual D-brane  $|\tilde{\lambda}\rangle_C$  of  $\widehat{\text{su}}(K)_N$ , is

$$\tilde{Q}_{\tilde{\lambda}} = (-1)^{r(\lambda)} Q_\lambda \pmod{x}, \quad \text{for } N + K \text{ odd.} \quad (4.1)$$

To establish eq. (4.1), we need to prove

$$(\dim \tilde{\lambda})_{\text{su}(K)} = (-1)^{r(\lambda)} (\dim \lambda)_{\text{su}(N)} \pmod{x}, \quad \text{for } N + K \text{ odd} \quad (4.2)$$

for any integrable highest-weight representation of  $\widehat{\text{su}}(N)_K$ .

*Proof:* Consider the branching of a representation  $\lambda$  of  $\text{su}(M_1 + M_2)$  into representations  $(\lambda', \lambda'')$  of  $\text{su}(M_1) \times \text{su}(M_2)$

$$(\dim \lambda)_{\text{su}(M_1 + M_2)} = \sum_{\lambda', \lambda''} \overline{N}_{\lambda' \lambda''}^\lambda (\dim \lambda')_{\text{su}(M_1)} (\dim \lambda'')_{\text{su}(M_2)}$$

where the integers  $\overline{N}_{\lambda' \lambda''}^\lambda$  are the Littlewood-Richardson coefficients [35]. Formally, we let  $M_1 = N + K$  and  $M_2 = -K$ , and then use

$$(\dim \lambda'')_{\text{su}(-K)} = (-1)^{r(\lambda'')} (\dim \tilde{\lambda}'')_{\text{su}(K)} \quad (4.3)$$

to obtain

$$(\dim \lambda)_{\text{su}(N)} = (-1)^{r(\lambda)} (\dim \tilde{\lambda})_{\text{su}(K)} + \sum_{\substack{\lambda' \neq 0 \\ \lambda''}} (-1)^{r(\lambda'')} \overline{N}_{\lambda' \lambda''}^\lambda (\dim \lambda')_{\text{su}(N+K)} (\dim \tilde{\lambda}'')_{\text{su}(K)} \quad (4.4)$$

where we have separated the  $\lambda' = 0$  term from the rest of the sum.

First consider the case where  $N + K$  is prime. Using the hook length formula for the dimension of a representation

$$(\dim \lambda')_{\text{su}(N+K)} = \prod_{(i,j)} \frac{N + K + j - i}{h_{ij}}, \quad (4.5)$$

where the product is over the boxes  $(i, j)$  of the Young tableau, labeled by their row  $i$  and column  $j$ , and  $h_{ij}$  are the corresponding hook lengths, we see that, for all  $\lambda' \neq 0$ , the numerator contains the factor  $N + K$ . The maximum hook length  $h_{11}$  is given by  $\ell_1 + k_1 - 1$ , where  $\ell_i$  and  $k_i$  are the row and column lengths respectively of the Young tableau. Since



$\lambda$  is an integrable representation of  $\widehat{\mathfrak{su}}(N)_K$ , its maximum hook length is  $N + K - 2$ , and therefore this is also true for  $\lambda'$ . Since none of the hook lengths divide  $N + K$  (prime), we have that  $(\dim \lambda')_{\mathfrak{su}(N+K)}$  is a multiple of  $N + K$  for all  $\lambda' \neq 0$ , thus the sum in eq. (4.4) is divisible by  $N + K$ , which establishes eq. (4.2) when  $N + K$  is prime, since  $x = N + K$  in this case.

For  $N + K$  not prime, we must use a different approach. The dimension of an arbitrary irreducible representation of  $\mathfrak{su}(N)$  can be written as the determinant of an  $\ell_1 \times \ell_1$  matrix [35]

$$\dim \lambda = \left| \dim \Lambda_{k_i+j-i} \right|, \quad i, j = 1, \dots, \ell_1 \quad (4.6)$$

where  $\Lambda_s$  is the completely antisymmetric representation of  $\mathfrak{su}(N)$ , whose Young tableau is  $\boxed{\begin{smallmatrix} & & \\ & & \\ & & \end{smallmatrix}}_s$ . The maximum value of  $s$  appearing in eq. (4.6) is  $k_1 + \ell_1 - 1$ , which is bounded by  $N + K - 2$  for integrable highest-weight representations of  $\widehat{\mathfrak{su}}(N)_K$ . For  $1 \leq s \leq N - 1$ ,  $\Lambda_s$  are the fundamental representations of  $\mathfrak{su}(N)$ , with  $\dim \Lambda_s = \binom{N}{s}$ . The representations  $\Lambda_0$  and  $\Lambda_N$  both correspond to the identity representation with dimension 1. We define  $\dim \Lambda_s = 0$  for  $s < 0$  and for  $s > N$ .

There is an alternative formula for the dimension of a representation in terms of the determinant of a  $k_1 \times k_1$  matrix, *viz.*, [35]

$$\dim \lambda = \left| \dim \tilde{\Lambda}_{\ell_i+j-i} \right|, \quad i, j = 1, \dots, k_1 \quad (4.7)$$

where  $\tilde{\Lambda}_s$  is the completely symmetric representation of  $\mathfrak{su}(N)$ , whose Young tableau is  $\underbrace{\boxed{\begin{smallmatrix} & & \\ & & \\ & & \end{smallmatrix}}}_s$ .

For  $s \geq 0$ ,  $\dim \tilde{\Lambda}_s = \binom{N+s-1}{s}$ . We define  $\dim \tilde{\Lambda}_s = 0$  for  $s < 0$ .

To establish level-rank duality of D-brane charges using eqs. (4.6) and (4.7), we need to obtain the relation between  $(\dim \Lambda_s)_{\mathfrak{su}(N)}$  and  $(\dim \tilde{\Lambda}_s)_{\mathfrak{su}(K)}$  for all  $s \leq N + K - 2$ . We consider three cases separately:

- $s \leq N - 1$ :

For the fundamental representations of  $\mathfrak{su}(N)$ , we use eq. (4.4) to establish that

$$(\dim \Lambda_s)_{\mathfrak{su}(N)} = (-1)^s (\dim \tilde{\Lambda}_s)_{\mathfrak{su}(K)} \pmod{x}, \quad s \leq N - 1 \quad (4.8)$$

since each term in the sum on the r.h.s. of eq. (4.4) includes a factor  $(\dim \Lambda_t)_{\mathfrak{su}(N+K)} = \binom{N+K}{t}$ ,  $1 \leq t \leq s$ , and

$$\gcd \left\{ \binom{N+K}{t} \mid t = 1, \dots, N - 1 \right\} = x, \quad (4.9)$$

as shown in Appendix C of ref. [14]. Equation (4.8) also applies trivially when  $s \leq 0$ .

- $N + 1 \leq s \leq N + K - 2$ :

For this case,  $(\dim \Lambda_s)_{\mathfrak{su}(N)}$  vanishes, so eq. (4.8) will hold provided that  $(\dim \tilde{\Lambda}_s)_{\mathfrak{su}(K)}$  vanishes mod  $x$ . To show this, we repeatedly use the identity  $\binom{M+1}{j} = \binom{M}{j} + \binom{M}{j-1}$  to show that

$$(\dim \tilde{\Lambda}_s)_{\mathfrak{su}(K)} = \binom{K+s-1}{s} = \sum_{u=0}^{s-N-1} \binom{s-N-1}{u} \binom{N+K}{N+u+1}, \quad N+1 \leq s \leq N+K-2 \quad (4.10)$$

Since  $\binom{N+K}{N+u+1} = \binom{N+K}{K-u-1}$ , and recalling that  $K \leq N$ , we see that each term in the sum in eq. (4.10) contains a factor belonging to the set in eq. (4.9), and therefore  $(\dim \tilde{\Lambda}_s)_{\text{su}(K)}$  is a multiple of  $x$ .

- $s = N$ :

The remaining case is easily evaluated using eq. (3.11):

$$\begin{aligned} (\dim \Lambda_N)_{\text{su}(N)} &= (\dim 0)_{\text{su}(N)} = (\dim 0)_{\text{su}(K)} \\ &= (-1)^{K-1} (\dim \sigma(0))_{\text{su}(K)} \mod x \\ &= (-1)^{N-K-1} [(-1)^N (\dim \tilde{\Lambda}_N)_{\text{su}(K)}] \mod x \end{aligned} \quad (4.11)$$

which is in accord with eq. (4.8), but *only* when  $N + K$  is odd.

To summarize, we have shown that

$$(\dim \Lambda_s)_{\text{su}(N)} = (-1)^s (\dim \tilde{\Lambda}_s)_{\text{su}(K)} \mod x, \quad s \leq N + K - 2, \quad \text{for } N + K \text{ odd.} \quad (4.12)$$

The equality (4.12) also holds for  $N + K$  even, except when  $s = N$ , in which case the sign is reversed.

Restricting ourselves to  $N + K$  odd, we use eq. (4.12) in eq. (4.6) to find

$$\begin{aligned} (\dim \lambda)_{\text{su}(N)} &= \left| (-1)^{k_i+j-i} (\dim \tilde{\Lambda}_{k_i+j-i})_{\text{su}(K)} \right| \mod x \\ &= (-1)^{r(\lambda)} \left| (\dim \tilde{\Lambda}_{k_i+j-i})_{\text{su}(K)} \right| \mod x, \quad \text{for } N + K \text{ odd} \end{aligned} \quad (4.13)$$

where  $r(\lambda) = \sum_{i=1}^{\ell_1} k_i(\lambda)$ . Comparing this with eq. (4.7), we see that (provided  $\ell_1 < K$ ) the r.h.s. is the dimension of a representation with row lengths  $k_i$  and column lengths  $\ell_i$ , that is, the transpose representation  $\tilde{\lambda}$ , hence

$$(\dim \lambda)_{\text{su}(N)} = (-1)^{r(\lambda)} (\dim \tilde{\lambda})_{\text{su}(K)} \mod x, \quad \text{for } N + K \text{ odd} \quad (4.14)$$

from which follows the level-rank duality of D-brane charges (4.1). One subtlety remains: if  $\ell_1 = K$  for  $\lambda$ , then the transpose  $\tilde{\lambda}$  contains leading columns of  $K$  boxes. In that case, one can use eq. (3.11)  $k_K$  times to relate  $\lambda$  to a tableau  $\hat{\lambda}$  with no rows of length  $K$ , and then apply eq. (4.14). The overall prefactor is then  $(-1)^{r(\hat{\lambda})+(N-1)k_K}$ , which is equal to  $(-1)^{r(\tilde{\lambda})+Kk_K} = (-1)^{r(\lambda)}$  since  $N + K$  is odd, so that eq. (4.14) holds in this case as well. *QED.*

The failure of eq. (4.12) to hold for  $s = N$  when  $N + K$  is even, however, precludes (4.14) from holding generally in this case. The adjoint of  $\text{SU}(3)_3$  provides a simple counterexample. If, however, the maximum hook length of  $\lambda$  (*viz.*,  $\ell_1 + k_1 - 1$ ) is less than  $N$ , then eq. (4.14) also holds for  $N + K$  even.

At present,<sup>4</sup> we do not know the precise relation between the charges of level-rank dual D-branes for even  $N + K$  when the maximum hook length is equal to or greater than  $N$ . (In many cases where  $N + K$  is even and  $N \neq K$ , however,  $x$  is unity, so eq. (4.1) is trivially satisfied.)

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<sup>4</sup>See footnote 3.

## Level-rank duality of the charge algebra

The level-rank duality of D-brane charges proved above, together with the previously-known level-rank duality for fusion coefficients, can be used to show that the charge algebras associated with  $\widehat{\mathfrak{su}}(N)_K$  and  $\widehat{\mathfrak{su}}(K)_N$  are isomorphic, in the case that  $N + K$  is odd.

We begin with the charge algebra for  $\widehat{\mathfrak{su}}(N)_K$

$$Q_\mu \cdot Q_\nu = \sum_{\lambda \in P_+^K} N_{\mu\nu}{}^\lambda Q_\lambda \quad (4.15)$$

or

$$(\dim \mu)_{\mathfrak{su}(N)} (\dim \nu)_{\mathfrak{su}(N)} = \sum_{\lambda \in P_+^K} N_{\mu\nu}{}^\lambda (\dim \lambda)_{\mathfrak{su}(N)} \pmod{x}. \quad (4.16)$$

Using eq. (4.14), we have

$$(\dim \tilde{\mu})_{\mathfrak{su}(K)} (\dim \tilde{\nu})_{\mathfrak{su}(K)} = \sum_{\lambda \in P_+^K} (-1)^{N\Delta} N_{\mu\nu}{}^\lambda (\dim \tilde{\lambda})_{\mathfrak{su}(K)} \pmod{x}, \quad \text{for } N + K \text{ odd} \quad (4.17)$$

where  $\Delta = [r(\mu) + r(\nu) - r(\lambda)] / N \in \mathbb{Z}$ . Since  $N + K$  is odd, we have  $(-1)^{N\Delta} = (-1)^{(K-1)\Delta}$ , which together with eq. (3.11) yields

$$(\dim \tilde{\mu})_{\mathfrak{su}(K)} (\dim \tilde{\nu})_{\mathfrak{su}(K)} = \sum_{\lambda \in P_+^K} N_{\mu\nu}{}^\lambda (\dim \sigma^\Delta(\tilde{\lambda}))_{\mathfrak{su}(K)} \pmod{x}, \quad \text{for } N + K \text{ odd}. \quad (4.18)$$

Finally, we use the level-rank duality of fusion coefficients (2.13) to obtain the level-rank dual charge algebra

$$(\dim \tilde{\mu})_{\mathfrak{su}(K)} (\dim \tilde{\nu})_{\mathfrak{su}(K)} = \sum_{\lambda \in P_+^K} \tilde{N}_{\tilde{\mu}\tilde{\nu}}{}^{\sigma^\Delta(\tilde{\lambda})} (\dim \sigma^\Delta(\tilde{\lambda}))_{\mathfrak{su}(K)} \pmod{x}, \quad \text{for } N + K \text{ odd}. \quad (4.19)$$

That the D-brane charges of  $\widehat{\mathfrak{su}}(K)_N$  satisfy the  $\widehat{\mathfrak{su}}(K)_N$  charge-algebra

$$\tilde{Q}_{\tilde{\mu}} \cdot \tilde{Q}_{\tilde{\nu}} = \sum_{\lambda \in P_+^K} \tilde{N}_{\tilde{\mu}\tilde{\nu}}{}^{\sigma^\Delta(\tilde{\lambda})} \tilde{Q}_{\sigma^\Delta(\tilde{\lambda})} \quad (4.20)$$

is obvious, but what we have shown is that the solutions to the  $\widehat{\mathfrak{su}}(N)_K$  charge algebra are isomorphic (modulo  $x$ ) to the solutions to the charge algebra of the dual D-branes in the  $\widehat{\mathfrak{su}}(K)_N$  theory, when  $N + K$  is odd. That is, term-by-term, the right hand sides of eqs. (4.15) and (4.20) match: each term  $Q_\lambda$  is equal (mod  $x$ ) to the corresponding term  $\tilde{Q}_{\sigma^\Delta(\tilde{\lambda})}$ , possibly modulo a sign common to all the terms in the sum.

Note that this isomorphism does not hold for  $N + K$  even. A simple counterexample is  $\square \otimes \square = 1 \oplus \blacksquare$  in  $\text{SU}(2)_2$ , where  $\dim(\blacksquare) = 3 \pmod{4}$  is not equal to  $\dim(\mathfrak{g}) = 1 \pmod{4}$ .

## Level-rank duality of the masses of Cardy states

In ref. [14], it is shown that the mass of a boundary state, normalized to that of the identity representation, is given by

$$\frac{\text{Energy}(|\lambda\rangle\rangle_C)}{\text{Energy}(|0\rangle\rangle_C)} = \frac{S_{0\lambda}}{S_{00}} \quad (4.21)$$

where the r.h.s. of this equation is simply the  $q$ -dimension of  $\lambda$  where  $q = e^{2\pi i/(N+K)}$ . It was shown in refs. [27] that  $q$ -dimensions of primary fields are invariant under level-rank duality

$$\left(\frac{S_{0\lambda}}{S_{00}}\right)_{\widehat{\mathfrak{su}}(N)_K} = \left(\frac{\tilde{S}_{0\tilde{\lambda}}}{\tilde{S}_{00}}\right)_{\widehat{\mathfrak{su}}(K)_N} \quad (4.22)$$

hence the masses of the level-rank dual Cardy states are equal:

$$\left(\frac{\text{Energy}(|\lambda\rangle_C)}{\text{Energy}(|0\rangle_C)}\right)_{\widehat{\mathfrak{su}}(N)_K} = \left(\frac{\text{Energy}(|\tilde{\lambda}\rangle_C)}{\text{Energy}(|0\rangle_C)}\right)_{\widehat{\mathfrak{su}}(K)_N}. \quad (4.23)$$

## 5 Conclusions

In this paper, we have begun to analyze the consequences of level-rank duality in boundary WZW models. We have found that the D0-charge of a symmetry-preserving D-brane  $|\lambda\rangle_C$  of the  $\widehat{\mathfrak{su}}(N)_K$  model is equal (up to a sign  $(-1)^{r(\lambda)}$ , where  $r(\lambda)$  is the number of boxes of the Young tableau of  $\lambda$ ) to the charge of the level-rank dual D-brane  $|\tilde{\lambda}\rangle_C$  of the  $\widehat{\mathfrak{su}}(K)_N$  model, provided that  $N + K$  is odd. (A similar relation for even  $N + K$ , but with a more complicated expression for the relative sign, is anticipated.<sup>5</sup>) The charges of D-branes are only defined modulo  $x_{N,K}$ , given by eq. (1.2), which is related to the twisted K-theory group of  $\widehat{\mathfrak{su}}(N)_K$ . Since the periodicities  $x_{N,K}$  are not level-rank dual, the charges of level-rank dual D-branes can only be compared modulo the periodicity of the smaller charge group.

We have also shown that the charge algebras of D-branes of level-rank dual theories are isomorphic, again when  $N + K$  is odd. Finally, we observed that the masses of the level-rank dual D-branes are equal, which follows from the level-rank duality of quantum dimensions of integrable highest weight representations.

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<sup>5</sup>See footnote 3.

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